Dynamic Transposition of Just Intonation Scale

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Abstract

A method is proposed which enables one to produce musical compositions by using transposition in place of harmonic progression. A transposition scale is introduced to provide a set of intervals commensurate with the musical scale, such as chromatic or just intonation scales. A sequence of intervals selected from this transposition scale is used to transpose instrument frequency at predefined times during the composition, which serves as a harmonic sequence of a composition. A harmonic sequence constructed in such a way can be extended to a hierarchy of harmonic sequences. In this case the fundamental sound frequency of an instrument is obtained as a product of the base frequency, instrument key factor, and a cumulative product of respective factors from all the harmonic sequences. The multiplication factors are selected from subsets of rational numbers, which form instrument scales and transposition scales of different levels. Each harmonic sequence can be related to its own transposition scale, or a single scale can be used for all levels. When composing for an orchestra of instruments, harmonic sequences and instrument scales can be assigned independently to each musical instrument. The method solves the problem of using just intonation scale across multiple octaves as well as simplifies writing of instrument scores.

Keywords

Just intonation scale, modulation, transposition, computer music.

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1 Introduction

Repetition is one of the main elements of music. Indeed, a common techniques used in composition is a repetition of a melodic sequence in different keys. The change of keys introduces another important element of music, that of gradual change (harmonic/chord progression or sequence), which removes the monotony of a simple repetition. In principle, the change of key in harmonic progression amounts to a frequency shift and can be done by transposition. For example, the technique of using barre chords when playing a guitar does exactly that, and this greatly enhances the versatility of the instrument.

To accomplish the same on a key-based instrument one can supply such instrument with extra transposition keys. This set of keys can be used during the performance to transpose the instrument each time the change of harmonic key needs to be done within the composition. In this case the melody can be played as if it always stays in the same key, which simplifies playing technique. Also, a number of keys in an octave of such instrument can be reduced to only the keys which form harmonic intervals, since most melodic sequences tend to comply to these intervals. For example, in chromatic scale one would use 8 tones, including major and minor triads, instead of 12 semitones.

There is another important advantage of such an instrument. Unlike a conventional keyboard or a guitar, it can be made to play in scales other than chromatic scale. One can easily assign the frequencies of a Pythagorean or just intonation scale to the keys of such instrument. It should be noted though that this kind of transposition cannot be easily accomplished on an acoustic instrument. For example, the barre technique on a guitar will fail because transposition in a just intonation scale will require simultaneous re-tuning of all the strings. This is because just intervals are not uniform across the octave as in chromatic scale. On the other hand, an electronic instrument or a computer sequencing application can easily accommodate such technique which we shall refer to as *dynamic transposition* and which is the main subject of this work.

2 Background

One limitation of a standard chromatic scale used in music today is the inexact representation of harmonic tones, as derived from the physics of resonating strings Wood and Bowsher (1980). Harmonic frequencies differ by rational multipliers, for example 2, 1/2, 3/2, 3/4, 4/3. Sounds combined of such frequencies are usually pleasant to the ear. This could be related to resonances caused by harmonic frequencies inside the ear, or some more complex phenomena inside the brain Levitin (2007); Cook (2001). Indeed, earlier instruments were based on resonating strings, and were tuned to follow these ratios which is reflected, for example, in Pythagorean scale Benward and Saker (2003). Here we will refer to a *rational scale* as a just intonation scale with intervals perceived as consonant. A significant limitation of a rational scale is that it is not *transpositionally invariant* in a sense that the same chords played in different keys will not sound the same (this definition differs somewhat from Milne et al. (2007)). This is because the intervals in a scale like 1, 6/5, 5/4, 4/3, 3/2, are not equal whereas in chromatic scale they are equally spaced on a logarithmic scale as is illustrated in Fig.1. This makes it impossible to play the same melody in different keys on a rational scale. Thus, a scale derived from rational numbers can not be easily applied to a key-based instrument, such as a piano, or an organ. Elaborate tuning systems and scales combined with novel keyboard concepts have been devised to overcome this limitation Fokker (1967, 1969); Fonville (1991); Milne et al. (2007), but these techniques were not exploring the extended use of transposition as proposed here, and actually added to the complexity of the keyboard. In a addition to that a variety of sound synthesis methods as well as spectral music techniques Anderson (2001) and frequency modulation synthesis Chowning (1973) have been developed, which focus more on acoustical quality of sound rather than on composition techniques and melodic sequencing, and are outside of the scope of this work.

A number of composers and performers experimented in just intonation scale, most notably, Harry Partch, James Tenney, and Ben Johnston. In particular, Harry Partch Gilmore (1998) did early experiments with tunings based on a larger number of unequal tones in the octave following just intonation and produced instruments which could play in this scale. James Tenney started one of the early and successful experimentations with electronic and computer generated music, in which he resorted to more harmoneous just intonation scales, and improved the acoustic quality of sound. The approach of Ben Johnson Johnston (2006) is based on extending the musical notation by introducing more intervals in line with just intonation scale. Their work demonstrated the advantages and drawbacks of musical instruments and composition techniques based on just intonation scale. However, the methods used did not rely on transposition as means of replacing harmonic progression in a sense the current work suggests.

Also, a number of novel tuning methods have been researched and proposed over the years Carlos (1987), including the asymmetric division of octave Sethares (1992) and adjusting the instrument tuning to timbre Sethares (2005). However, these techniques either increase the complexity of an instrument or like chromatic scale, they compromise the purity of intervals derived from the rational scale. Indeed, frequencies produced by chromatic scale no longer differ by rational multipliers, but rather by transcendental numbers resulting from the logarithmic operation as is evident from Fig.1. This is the reason why chords played on an acoustic piano sound more pleasantly than those played on an electronic keyboard, since the strings of an acoustic instrument can self-adjust their frequencies due to the resonance effect. This makes them lapse to more pleasant rational intervals Carlos (1987).

There are two main techniques of frequency shift: modulation and transposition. Modulation has been mainly used for sound effects, or as an occasional semi-tone shift

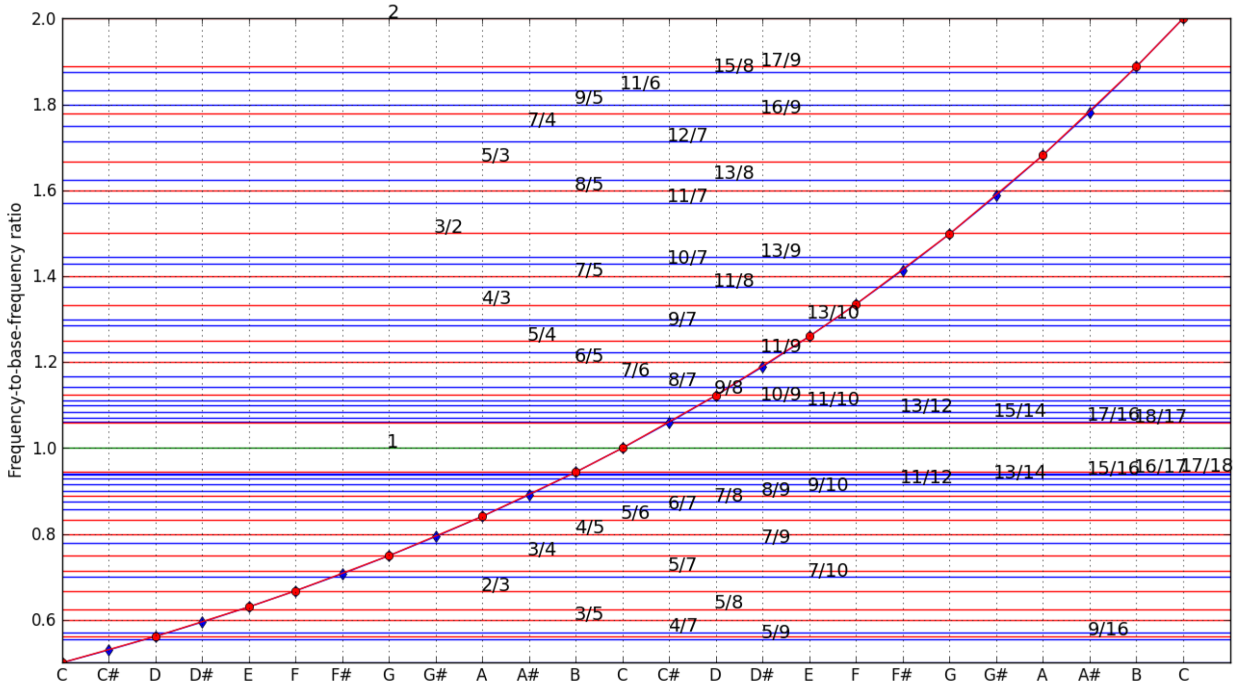

Figure 1. Frequencies of chromatic scale

Number above each line added to one is equal to the scaling factor used to obtain the frequency associated with that line. Red points correspond to the C-major scale.

in a composition. A wider frequency shift was mainly limited to octave shift in some electronic keyboards. Shifting of instrument's frequency to a different key is commonly done with transposition. However, this shift is usually made permanent for the duration of the performance. To author's knowledge neither modulation nor transposition have been widely used in place of harmonic progression, that is, to change current key in a composition. However, such application can enable the usage of rational scales, simplify a keyboard instrument, as well as reduce the number of notes in octave Smirnov (2013), tectral.com.

In particular, the number of tones in the octave, or cardinality, can be reduced to only the most harmonic musical intervals, such as those in a rational scale. This is because the number of keys in the octave can be made equal to the number of tones in the scale. For example, instead of selecting 7 keys for a chromatic scale out of 12 semitones in an octave, one is left with just 7 keys corresponding to 7 tones, thus no need for selection. And for simpler scales this number can be further reduced. Now, instead of a multitude of tonic combinations for every key one has a single scale, which is a set of basic ratios. For example, instead of memorizing major/minor tonic triads for every key, such as "C,E,G", "C,$^\flat$E,G" for major/minor triads in C-key, etc. (a total of 2·12=24 combinations), one will have to remember only two combinations of ratios: (5/4, 3/2) for a major triad and (6/5, 3/2) for a minor triad, the first number in the triad beign always 1. Shifting to different chords in a progression will amount to multiplication of these numbers by a rational number selected from the currently used scale. For example, a sequence (1, 9/8, 5/4, 4/3, 3/2, 5/6, 15/8, 2) will resemble the chromatic major scale.

We will refer to an octave setup according to the rational scale as *instrument scale*. In this case a harmonic progression can be replaced by a *transposition sequence*, which we will also refer to as a *harmonic sequence* for clarity. This harmonic sequence can be further generalized to a hierarchy of harmonic sequences of different levels. A harmonic sequence on each level of hierarchy is derived from a *transposition scale*, which is also a rational scale, i.e. a subset of rational numbers. Each transposition scale can be equal to the instrument scale, or can differ from it, thus creating a *multi-scale composition*. Each harmonic sequence can be related to its own scale, or a single scale can be used for all levels. Frequencies of sounds are obtained by multiplying the base frequency by the factor corresponding the current instrument key selected from the instrument scale and by a cumulative product of factors selected from harmonic sequences on all levels and corresponding to the current time interval in the composition.

One can also extend this method of multi-scale composition to a *multi-scale orchestra* of instruments. In this case when composing for an orchestra, harmonic sequences and scales can be assigned independently to each musical instrument. This method also opens up opportunities of exploring different musical scales that can exist within the realm of physical resonances.

3 Harmonic Sequences

In the proposed system of generating musical sounds the fundamental frequency of each sound is obtained as a product between the base frequency and a cumulative product of rational multipliers. Looking from a slightly different perspective, the fundamental frequency is obtained as a multiple of another frequency, which in turn can be obtained as a multiple of yet another frequency, and so on. The multipliers can be selected from a subset of rational numbers, defined by simple ratios of two integers.

In the simplest case, which we shall refer to as a *level-1* composition, the procedure of creating a composition starts with a single *base frequency* and a subset of rational numbers, further referred to as the *instrument scale*, and each number in the set will be referred to as the *instrument key*. Each sound in a composition is characterized by its fundamental frequency, further referred to as the *note frequency*, and the associated time interval. The note frequency is obtained by multiplying the base frequency by a factor taken from the instrument scale and corresponding to the current instrument key. We will refer to this number as the *scale key*. The pair consisting of the scale key and the corresponding time interval will be referred to as the *note*. Thus, the procedure of level-1 composition sets for each time interval in a composition a corresponding note frequency equal to a product of the base frequency and one of the keys selected from the instrument scale. This key selection can be done independently for each instrument. The time intervals can be overlapping, thus allowing for playing chords. This sequence of notes selected from the instrument scale will be referred to as the *instrument score*. The manner in which the instrument score is compiled is not important for this discussion and is presumably done by a composer.

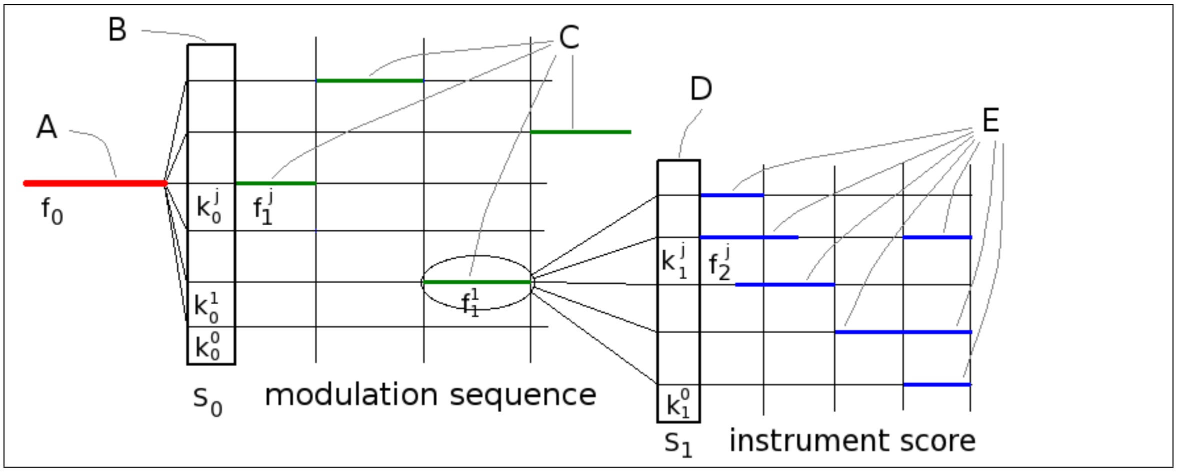

Figure 2. Assigning frequencies in level-2 composition: A - base frequency, B - transposition scale, C - harmonic sequence, D - instrument scale, E - instrument score.

In the next step this procedure is extended to *level-2* composition (Fig.2). In this case in addition to the base frequency (A), instrument scale (D), and instrument score (E), there is another layer of rational numbers further referred to as the *transposition scale* (B), inserted between the base frequency and the instrument scale. Similarly to the level-1 composition, a sequence of numbers is selected from this scale and assigned to the corresponding time intervals, thus resulting in another sequence of key-time pairs, which we will refer to as the *harmonic sequence* (C), and each pair in the sequence will be referred to as the *transposition tone*. Unlike the notes of the instrument score, time intervals of transposition tones should be non-overlapping and span the entire composition. Selection of transposition keys and associated time intervals to make up a harmonic sequence is done by a composer and it will determine the harmonic structure of the composition. In the notation of Fig.2 where f_i^j represents frequency on i-th level, obtained from j-th key in i-th scale, k_i^j. The instrument sound frequencies of level-2 composition (f_2^j) are determined as triple products of the base frequency, transposition tones, and instrument keys. The selection of a transposition tone for each note is always possible and unique, because the time intervals of transposition tones are restricted to be non-overlapping and to span the entire composition. The level-2 composition procedure corresponds to a conventional composition where the instrument score takes the place of a conventional score and harmonic sequence takes place of a chord progression (harmony of the song).

One can further generalize the above procedure to *level-n* composition through an n-step recursive frequency transformation which is specified for each time, t, in the composition, as:

$$f_{n+1}(t) = f_n(t)m_n(t) = f_0 \prod_{i=0}^{n} m_i(t) \tag{1}$$

where the initial frequency, f_0, or the *base frequency* will be a time-constant: $f_0(t) = f_0 = const$, and factors $m_n(t)$ for $n > 0$, correspond to transposition tones for time t of a harmonic sequence of level n, and $m_0(t)$ corresponds to the instrument score. Each $m_n(t)$ time sequence is selected from the corresponding n-level scale S_n as:

$$m_n(t) = \hat{\mathcal{R}}_n^{(t)} S_n \tag{2}$$

where $\hat{\mathcal{R}}_n^{(t)}$ denotes a generally time-dependent selection operator provided by a composer or an algorithm, and scale S_n on level n is an ordered collection of keys, k_n^i, represented by a subset of rational numbers, i.e.

$$S_n = \bigcup_j k_n^j$$

The above requirement of non-overlapping time intervals in a harmonic sequence means that only the 0-level selection operator, $\hat{\mathcal{R}}_0^{(t)}$, acting on the instrument scale in (2), is allowed to generate multiple selections for the same value of t, thereby enabling chords in the instrument score. Chords have no obvious meaning for a harmonic sequence as defined here.

For example, in level-3 composition we introduce a second harmonic sequence and optionally the associated transposition scale. The frequency of sound in this case will be determined as a product of the base frequency, the key from the currently played note of a particular instrument score, and the transposition tones from the harmonic sequences of levels 1 and 2, both corresponding to the current time interval. Thus, according to (1) the key note frequency in an instrument score will be determined as:

$$f_3 = f_2 m_2 = f_1 m_1 m_2 = f_0 m_0 m_1 m_2$$

In this case the instrument score is provided by sequence m_0, which corresponds to a conventional score, harmonic sequence m_1 will correspond to a harmonic structure of a song, such as a sequence of keys assigned to different measures, e.g. C, Am, F, G, and harmonic sequence m_2 can be related to a conventional transposition, which in this case will up/down-shift all frequencies in certain parts of the song. The distinction from the conventional composition will be in simplification in the instrument score, because the transition between different keys (chord progression) is already taken care of by the harmonic sequences. Another simplification comes from the lower cardinality of the octave as well as from the fact that the problem of selecting appropriate tones corresponding to a current melodic key no longer exists.

It should be noted that one can also use the reverse definition of sequences m_i, where the last sequence takes place of an instrument score and the rest perform the transposition, which corresponds to the illustration in Fig.2.

In the case of multiple instruments (orchestra) it is possible to assign different instrument scales to different instruments and even to direct them to follow different harmonic sequences. In the simplest case the base-level harmonic sequence may consist of a single tone, which will correspond to a transposition of that instrument to a different frequency range for a duration of the composition.

For human composers and performers it will probably be not very practical to go beyond level-3 composition. Nevertheless, higher level multi-scale compositions can be explored in algorithmically generated compositions.

4 Implementation

The principles outlined above can be easily implemented in a music sequencing software, by essentially adapting standard MIDI-sequencer techniques. Fig.3 shows a generic piano-roll type panel where piano keys are replaced with an ordered set of ratios, which can represent either instrument or transposition scales. This setup can be used for entering instrument scores as well as harmonic sequences - all in a similar manner. It should be noted, that the two octaves of numerical keys shown are all the keys needed to play an instrument. Shifting to higher or lower octaves can be done by harmonic sequencing.

Fig.4 shows a generic view of a song editor, where along with traditional instrument tracks a harmony track is added. In this case it provides a global harmonic sequence for all instruments.

It is also feasible to design an electronic musical instrument based on these

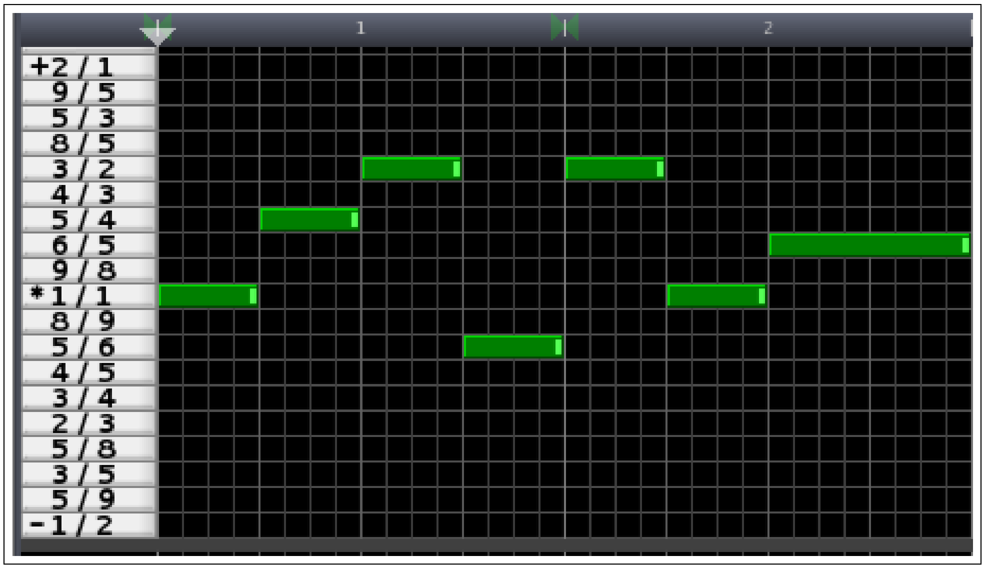

Figure 3. Instrument score or harmonic sequence in a "piano"-roll

principles. If implemented in a keyboard fashion, such instrument will require a small set of keys for a single octave or a couple of octaves only. The keys will correspond to a pre-defined instrument scale. A common 7-tone scale will suffice for most purposes, resulting in a 7-key keyboard. But even simpler 4-6 key instruments could be useful for playing basses or for less sophisticated game oriented devices. A transposition control for such instrument can be implemented in a separate key, knob, touch-pad, or similar. Whole harmonic sequences can be programmed-in for the purposes of performance. In this case switching between transposition tones in a sequence can be done automatically, or by pushing a separate key.

No attempt is made in this work to introduce a musical notation to accompany the proposed method. As long as the technique is not used for life performance no need for any specific notation arises. Indeed, in the current implementation all instrument scores and harmonic sequences are recorded and modified by means of a piano roll editor shown in Fig.3 and then stored in respective computer files. The final result is an audio file used for playback, while the human-readable scores are available either via editor interface of the respective sequencer application or in XML-encoded text files. The

Figure 4. Modulation sequence displayed as a track in a song editor

notation used in those files is similar to the MIDI standard modified for handling non-chromatic frequencies and harmonic sequences.

Nevertheless, a variety of specific musical notations can be developed for the purpose of life performance. This can be a simplification of the conventional notation. For example, the accidentals can be dropped since the cardinality of the octave is equal to the number of tones in the scale. Thus, all notes can be labeled by plain letters, including those of harmonic sequences. Numerical labeling by respective rational multiplies can be used for rigour or color labeling can be employed to enhance score readability in life performances. Ultimately, the notation used should depend on the key layout of the instrument so as to facilitate reading of the score. Thus, with color labeled keys the respective color labeling for the score would be appropriate.

Also, to simplify the performance, the harmonic sequence(s) can be stored and read by the instrument, while the performer will only read the instrument score. In this case each change of harmonic key in chord progression can be done either automatically by the instrument or manually by the performer, using a single key or a foot pedal.

5 Conclusions

As mentioned in the background section, the limitation of a just intonation scale is in its inability to achieve transpositional invariance. In the proposed method this limitation is overcome by introducing a generalized transposition as a system of multiple scales, and hierarchical harmonic sequences derived from the sets of rational numbers. In this framework it is now possible to replace harmonic progression with transposition of the base frequency to any value, and do so independently for different instruments. In this way one can play the same melodic sequence of tones in different keys and still realize a simple rational scaling of the base frequency as well as to retain the uniformity of key patterns within any harmonic key, or indeed within any frequency range. Thus, the proposed method enables the usage of musical scales beyond chromatic, including just intonation and other rational scales. From the perspective of physical reality of resonances and wave harmonics, frequencies produced as rational multiples of the fundamental frequency are more natural, and therefore tend to be more pleasant to human ear, which is indeed confirmed by old traditions and modern research.

The proposed method can be used for both composing music by means of sequencer applications and for life performances. It is mostly applicable to electronic instruments, computers, and other sound-capable digital devices. In addition to a more harmonious sound generation the method can simplify writing of musical scores as well as playing instruments built on these principles. The method also opens possibilities of exploring multi-level transposition for algorithmic composition.

References

Anderson, J. 2001. "Spectral Music." In S. Sadie, and J. Tyrrell, (editors) *The New Grove Dictionary of Music and Musicians. Second edition*. Macmillan Publishers. URL http://en.wikipedia.org/wiki/Spectral_music.

Benward, B., and M. N. Saker. 2003. *Music: In Theory and Practice*, volume 1. McGraw-Hill.

Carlos, W. 1987. “Tuning: At the Crossroads.” *Computer Music Journal* :29–43.

Chowning, J. 1973. “"The Synthesis of Complex Audio Spectra by Means of Frequency Modulation".” *Journal of the Audio Engineering Society* 21(7).

Cook, P. R. 2001. *Music, Cognition, and Computerized Sound: An Introduction to Psychoacoustics*. MIT Press.

Fokker, A. 1967. “On the expansion of the musician’s realm of harmony.” *Acta musicologica* 38:197–202.

Fokker, A. 1969. “Unison vectors and periodicity blocks in the three-dimensional (3-5-7-) harmonic lattice of notes.” In *Proceedings of the KNAW, B*, volume 72. pp. 153–168.

Fonville, J. 1991. “Ben Johnston’s Extended Just Intonation: A Guide for Interpreters.” *Perspectives of New Music* 29(2):121.

Gilmore, B. 1998. *"Harry Partch: A Biography"*. Yale University Press.

Johnston, B. 2006. *"Maximum Clarity" and Other Writings on Music*. Urbana: University of Illinois Press.

Levitin, D. J. 2007. *This is your brain on music. The science of human obsession*. Plume.

Milne, A., W. Sethares, and J. Plamondon. 2007. “Invariant Fingerings Across a Tuning Continuum.” *Computer Music Journal* 31:4:15–32.

Sethares, W. A. 1992. “Relating Tuning and Timbre.” *Experimental Musical Instruments* IX (2).

Sethares, W. A. 2005. *Tuning, Timbre, Spectrum, Scale*. Springer.

Smirnov, A. 2013. “Music Machine.” US Patent 8,541,677.

Wood, A., and J. M. Bowsher. 1980. *The physics of music*. Greenwood Press.